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NECESSARY OPTIMALITY CONDITIONS FOR FRACTIONAL DIFFERENCE PROBLEMS OF THE CALCULUS OF VARIATIONS

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Abstract. We introduce a discrete-time fractional calculus of variations. First and second order necessary optimality conditions are established. Examples illustrating the use of the new Euler-Lagrange and Legendre type conditions are given. They show that the solutions of the fractional problems coincide with the solutions of the corresponding non-fractional variational problems when the order of the discrete derivatives is an integer value.

1. **Introduction.** The Fractional Calculus is currently a very important research field in several different areas: physics (including classical and quantum mechanics and thermodynamics), chemistry, biology, economics and control theory [20, 23, 24, 27, 28]. It has origin more than 300 years ago when L'Hopital asked Leibniz what should be the meaning of a derivative of order 1/2. After that episode several more famous mathematicians contributed to the development of Fractional Calculus: Abel, Fourier, Liouville, Riemann, Riesz, just to mention a few names.

In [22] Miller and Ross define a fractional sum of order $\nu > 0$ *via* the solution of a linear difference equation. Namely, they present it as (see Section 2 for the notations used here)

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{(\nu-1)} f(s). \tag{1}$$

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This was done in analogy with the Riemann–Liouville fractional integral of order $\nu > 0$,

$$ {}_a\mathbf{D}_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-s)^{\nu-1} f(s) ds, $$

which can be obtained *via* the solution of a linear differential equation [22, 23]. Some basic properties of the sum in (1) were obtained in [22]. More recently, F. Atici and P. Eloe [9, 10] defined the fractional difference of order $\alpha > 0$, i.e., $\Delta^\alpha f(t) = \Delta^m(\Delta^{-(m-\alpha)} f(t))$ with m the least integer satisfying $m \geq \alpha$, and developed some of its properties that allow to obtain solutions of certain fractional difference equations.

Fractional differential calculus has been widely developed in the past few decades due mainly to its demonstrated applications in various fields of science and engineering. The study of fractional problems of the Calculus of Variations and respective Euler-Lagrange equations is a fairly recent issue – see [1, 2, 5, 7, 8, 11, 13, 15, 16, 21] and references therein – and include only the continuous case. It is well known that discrete analogues of differential equations can be very useful in applications [18, 19]. Therefore, we consider pertinent to start here a fractional discrete-time theory of the calculus of variations.

Our objective is two-fold. On one hand we proceed to develop the theory of *fractional difference calculus*, namely, we introduce the concept of left and right fractional sum/difference (cf. Definitions 2.1 and 2.4 below) and prove some new results related to them. On the other hand, we believe that the present work will potentiate research not only in the fractional calculus of variations but also in solving fractional difference equations, specifically, fractional equations in which left and right fractional differences appear.

Because the theory of fractional difference calculus is in its infancy [9, 10, 22], the paper is self contained. We begin, in Section 2, to give the definitions and results needed throughout. In Section 3 we present and prove the new results; in Section 4 we give some examples. Finally, in Section 5 we mention the main conclusions of the paper, and some possible extensions and open questions. Computer code done in the Computer Algebra System Maxima is given in Appendix.

2. **Preliminaries.** We begin by introducing some notation used throughout. Let a be an arbitrary real number and $b = k + a$ for a certain $k \in \mathbb{N}$ with $k \geq 2$. We put $\mathbb{T} = \{a, a+1, \ldots, b\}$, $\mathbb{T}^\kappa = \{a, a+1, \ldots, b-1\}$ and $\mathbb{T}^{\kappa^2} = \{a, \ldots, b-2\}$. Denote by $\mathcal{F}$ the set of all real valued functions defined on $\mathbb{T}$. Also, we will frequently write $\sigma(t) = t+1$, $\rho(t) = t-1$ and $f^\sigma(t) = f(\sigma(t))$. The usual conventions $\sum_{t=c}^{c-1} f(t) = 0$, $c \in \mathbb{T}$, and $\prod_{i=0}^{-1} f(i) = 1$ remain valid here.

As usual, the forward difference is defined by $\Delta f(t) = f^\sigma(t) - f(t)$. If we have a function f of two variables, $f(t,s)$, its partial (difference) derivatives are denoted by Δ_t and Δ_s, respectively. For arbitrary $x, y \in \mathbb{R}$ define (when it makes sense)

$$ x^{(y)} = \frac{\Gamma(x+1)}{\Gamma(x+1-y)}, $$

where Γ is the gamma function. The following property of the gamma function,

$$ \Gamma(x+1) = x\Gamma(x), \tag{2} $$

will be frequently used.

As was mentioned in Section 1, equality (1) was introduced in [22] as *the fractional sum of order* $\nu > 0$. While reaching the proof of Theorem 3.2 we actually "find" the definition of left and right fractional sum:

Definition 2.1. Let $f \in \mathcal{F}$. The *left fractional sum* and the *right fractional sum* of order $\nu > 0$ are defined, respectively, as

$$ {}_a\Delta_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{(\nu-1)} f(s), \tag{3} $$

and

$$ {}_t\Delta_b^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=t+\nu}^{b} (s-\sigma(t))^{(\nu-1)} f(s). \tag{4} $$

Remark 1. The above sums (3) and (4) are defined for $t \in \{a+\nu, a+\nu+1, \ldots, b+\nu\}$ and $t \in \{a-\nu, a-\nu+1, \ldots, b-\nu\}$, respectively, while $f(t)$ is defined for $t \in \{a, a+1, \ldots, b\}$. Throughout we will write (3) and (4), respectively, in the following way:

$$ {}_a\Delta_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t} (t+\nu-\sigma(s))^{(\nu-1)} f(s), \quad t \in \mathbb{T}, $$

$$ {}_t\Delta_b^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=t}^{b} (s+\nu-\sigma(t))^{(\nu-1)} f(s), \quad t \in \mathbb{T}. $$

Remark 2. The left fractional sum defined in (3) coincides with the fractional sum defined in [22] (see also (1)). The analogy of (3) and (4) with the Riemann–Liouville left and right fractional integrals of order $\nu > 0$ is clear:

$$ {}_a\mathbf{D}_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-s)^{\nu-1} f(s) ds, $$

$$ {}_x\mathbf{D}_b^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (s-x)^{\nu-1} f(s) ds. $$

It was proved in [22] that $\lim_{\nu\to 0} {}_a\Delta_t^{-\nu} f(t) = f(t)$. We do the same for the right fractional sum using a different method. Let $\nu > 0$ be arbitrary. Then,

$$ \begin{aligned} {}_t\Delta_b^{-\nu} f(t) &= \frac{1}{\Gamma(\nu)} \sum_{s=t}^{b} (s+\nu-\sigma(t))^{(\nu-1)} f(s) \\ &= f(t) + \frac{1}{\Gamma(\nu)} \sum_{s=\sigma(t)}^{b} (s+\nu-\sigma(t))^{(\nu-1)} f(s) \\ &= f(t) + \sum_{s=\sigma(t)}^{b} \frac{\Gamma(s+\nu-t)}{\Gamma(\nu)\Gamma(s-t+1)} f(s) \\ &= f(t) + \sum_{s=\sigma(t)}^{b} \frac{\prod_{i=0}^{s-t-1}(\nu+i)}{\Gamma(s-t+1)} f(s). \end{aligned} $$

Therefore, $\lim_{\nu\to 0} {}_t\Delta_b^{-\nu} f(t) = f(t)$. It is now natural to define

$$ {}_a\Delta_t^{0} f(t) = {}_t\Delta_b^{0} f(t) = f(t), \tag{5} $$

which we do here, and to write

$$ {}_a\Delta_t^{-\nu} f(t) = f(t) + \frac{\nu}{\Gamma(\nu+1)} \sum_{s=a}^{t-1} (t+\nu-\sigma(s))^{(\nu-1)} f(s), \quad t \in \mathbb{T}, \quad \nu \geq 0, \tag{6} $$

$$ {}_t\Delta_b^{-\nu} f(t) = f(t) + \frac{\nu}{\Gamma(\nu+1)} \sum_{s=\sigma(t)}^{b} (s+\nu-\sigma(t))^{(\nu-1)} f(s), \quad t \in \mathbb{T}, \quad \nu \geq 0. $$

The next theorem was proved in [10].

Theorem 2.2 ([10]). *Let $f \in \mathcal{F}$ and $\nu > 0$. Then, the equality*

$$ {}_a\Delta_t^{-\nu} \Delta f(t) = \Delta({}_a\Delta_t^{-\nu} f(t)) - \frac{(t+\nu-a)^{(\nu-1)}}{\Gamma(\nu)} f(a), \quad t \in \mathbb{T}^\kappa, $$

holds.

Remark 3. It is easy to include the case $\nu = 0$ in Theorem 2.2. Indeed, in view of (2) and (5), we get

$$ {}_a\Delta_t^{-\nu} \Delta f(t) = \Delta({}_a\Delta_t^{-\nu} f(t)) - \frac{\nu}{\Gamma(\nu+1)} (t+\nu-a)^{(\nu-1)} f(a), \quad t \in \mathbb{T}^\kappa, \tag{7} $$

for all $\nu \geq 0$.

Now, we prove the counterpart of Theorem 2.2 for the right fractional sum.

Theorem 2.3. *Let $f \in \mathcal{F}$ and $\nu \geq 0$. Then, the equality*

$$ {}_t\Delta_{\rho(b)}^{-\nu} \Delta f(t) = \frac{\nu}{\Gamma(\nu+1)} (b+\nu-\sigma(t))^{(\nu-1)} f(b) + \Delta({}_t\Delta_b^{-\nu} f(t)), \quad t \in \mathbb{T}^\kappa, \tag{8} $$

holds.

Proof. We only prove the case $\nu > 0$ as the case $\nu = 0$ is trivial (see Remark 3). We start by fixing an arbitrary $t \in \mathbb{T}^\kappa$. Then, we have that, for all $s \in \mathbb{T}^\kappa$,

$$ \begin{aligned} &\Delta_s \left((s+\nu-\sigma(t))^{(\nu-1)} f(s)) \right) \\ &\qquad\qquad = (\nu-1)(s+\nu-\sigma(t))^{(\nu-2)} f^\sigma(s) + (s+\nu-\sigma(t))^{(\nu-1)} \Delta f(s), \end{aligned} $$

hence,

$$ \begin{aligned} &\frac{1}{\Gamma(\nu)} \sum_{s=t}^{b-1} (s+\nu-\sigma(t))^{(\nu-1)} \Delta f(s) \\ &\quad = \left[\frac{(s+\nu-\sigma(t))^{(\nu-1)}}{\Gamma(\nu)} f(s) \right]_{s=t}^{s=b} - \frac{1}{\Gamma(\nu)} \sum_{s=t}^{b-1} (\nu-1)(s+\nu-\sigma(t))^{(\nu-2)} f^\sigma(s) \\ &\quad = \frac{(b+\nu-\sigma(t))^{(\nu-1)}}{\Gamma(\nu)} f(b) - \frac{(\nu-1)^{(\nu-1)}}{\Gamma(\nu)} f(t) \\ &\qquad - \frac{1}{\Gamma(\nu)} \sum_{s=t}^{b-1} (\nu-1)(s+\nu-\sigma(t))^{(\nu-2)} f^\sigma(s). \end{aligned} $$

We now compute $\Delta({}_t\Delta_b^{-\nu} f(t))$:

$$\begin{aligned}\Delta({}_t\Delta_b^{-\nu} f(t)) &= \frac{1}{\Gamma(\nu)}\left[\sum_{s=\sigma(t)}^{b}(s+\nu-\sigma(t+1))^{(\nu-1)}f(s)\right.\\ &\qquad\left.-\sum_{s=t}^{b}(s+\nu-\sigma(t))^{(\nu-1)}f(s)\right]\\ &= \frac{1}{\Gamma(\nu)}\left[\sum_{s=\sigma(t)}^{b}(s+\nu-\sigma(t+1))^{(\nu-1)}f(s)\right.\\ &\qquad\left.-\sum_{s=\sigma(t)}^{b}(s+\nu-\sigma(t))^{(\nu-1)}f(s)\right]-\frac{(\nu-1)^{(\nu-1)}}{\Gamma(\nu)}f(t)\\ &= \frac{1}{\Gamma(\nu)}\sum_{s=\sigma(t)}^{b}\Delta_t(s+\nu-\sigma(t))^{(\nu-1)}f(s)-\frac{(\nu-1)^{(\nu-1)}}{\Gamma(\nu)}f(t)\\ &= -\frac{1}{\Gamma(\nu)}\sum_{s=t}^{b-1}(\nu-1)(s+\nu-\sigma(t))^{(\nu-2)}f^{\sigma}(s)-\frac{(\nu-1)^{(\nu-1)}}{\Gamma(\nu)}f(t).\end{aligned}$$

Since t is arbitrary, the theorem is proved. □

Definition 2.4. Let $0<\alpha\leq 1$ and set $\mu=1-\alpha$. Then, the *left fractional difference* and the *right fractional difference* of order α of a function $f\in\mathcal{F}$ are defined, respectively, by

$${}_a\Delta_t^{\alpha} f(t)=\Delta({}_a\Delta_t^{-\mu} f(t)),\quad t\in\mathbb{T}^{\kappa},$$

and

$${}_t\Delta_b^{\alpha} f(t)=-\Delta({}_t\Delta_b^{-\mu} f(t)),\quad t\in\mathbb{T}^{\kappa}.$$

3. **Main Results.** Our aim is to introduce the discrete-time fractional problem of the calculus of variations and to prove corresponding necessary optimality conditions. In order to obtain an analogue of the Euler-Lagrange equation (cf. Theorem 3.5) we first prove a fractional formula of summation by parts. The results of the paper give discrete analogues to the fractional Riemann–Liouville results available in the literature: Theorem 3.2 is the discrete analog of fractional integration by parts [26, 27]; Theorem 3.5 is the discrete analog of the fractional Euler-Lagrange equation of Agrawal [1, Theorem 1]; the natural boundary conditions (22) and (23) are the discrete fractional analogues of the transversality conditions in [3, 21]. However, to the best of the authors knowledge, no counterpart to our Theorem 3.6 exists in the literature of continuous fractional variational problems.

3.1. **Fractional Summation by Parts.** The next lemma is used in the proof of Theorem 3.2.

Lemma 3.1. *Let f and h be two functions defined on $\mathbb{T}^{\kappa}$ and g a function defined on $\mathbb{T}^{\kappa}\times\mathbb{T}^{\kappa}$. Then, the equality*

$$\sum_{\tau=a}^{b-1} f(\tau)\sum_{s=a}^{\tau-1} g(\tau,s)h(s)=\sum_{\tau=a}^{b-2} h(\tau)\sum_{s=\sigma(\tau)}^{b-1} g(s,\tau)f(s)$$

holds.

Proof. Choose $\mathbb{T} = \mathbb{Z}$ and $F(\tau, s) = f(\tau)g(\tau, s)h(s)$ in Theorem 10 of [6]. □

The next result gives a *fractional summation by parts* formula.

Theorem 3.2 (Fractional summation by parts)**.** *Let f and g be real valued functions defined on $\mathbb{T}^k$ and $\mathbb{T}$, respectively. Fix $0 < \alpha \leq 1$ and put $\mu = 1 - \alpha$. Then,*

$$\begin{aligned}\sum_{t=a}^{b-1} f(t)_a\Delta_t^\alpha g(t) &= f(b-1)g(b) - f(a)g(a) + \sum_{t=a}^{b-2} {}_t\Delta_{\rho(b)}^\alpha f(t) g^\sigma(t)\\ &+ \frac{\mu}{\Gamma(\mu+1)} g(a) \left(\sum_{t=a}^{b-1} (t+\mu-a)^{(\mu-1)} f(t) - \sum_{t=\sigma(a)}^{b-1} (t+\mu-\sigma(a))^{(\mu-1)} f(t) \right).\end{aligned}$$

Proof. From (7) we can write

$$\begin{aligned}\sum_{t=a}^{b-1} f(t)_a\Delta_t^\alpha g(t) &= \sum_{t=a}^{b-1} f(t)\Delta({}_a\Delta_t^{-\mu} g(t))\\ &= \sum_{t=a}^{b-1} f(t) \left[{}_a\Delta_t^{-\mu}\Delta g(t) + \frac{\mu}{\Gamma(\mu+1)} (t+\mu-a)^{(\mu-1)} g(a) \right]\\ &= \sum_{t=a}^{b-1} f(t)_a\Delta_t^{-\mu}\Delta g(t) + \sum_{t=a}^{b-1} \frac{\mu}{\Gamma(\mu+1)} (t+\mu-a)^{(\mu-1)} f(t) g(a).\end{aligned} \tag{9}$$

Using (6) we get

$$\begin{aligned}&\sum_{t=a}^{b-1} f(t)_a\Delta_t^{-\mu}\Delta g(t)\\ &= \sum_{t=a}^{b-1} f(t)\Delta g(t) + \frac{\mu}{\Gamma(\mu+1)} \sum_{t=a}^{b-1} f(t) \sum_{s=a}^{t-1} (t+\mu-\sigma(s))^{(\mu-1)} \Delta g(s)\\ &= \sum_{t=a}^{b-1} f(t)\Delta g(t) + \frac{\mu}{\Gamma(\mu+1)} \sum_{t=a}^{b-2} \Delta g(t) \sum_{s=\sigma(t)}^{b-1} (s+\mu-\sigma(t))^{(\mu-1)} f(s)\\ &= f(b-1)[g(b) - g(b-1)] + \sum_{t=a}^{b-2} \Delta g(t)_t\Delta_{\rho(b)}^{-\mu} f(t),\end{aligned}$$

where the third equality follows by Lemma 3.1. We proceed to develop the right hand side of the last equality as follows:

$$\begin{aligned}
&f(b-1)[g(b)-g(b-1)]+\sum_{t=a}^{b-2}\Delta g(t)_t\Delta_{\rho(b)}^{-\mu}f(t)\\
&\quad=f(b-1)[g(b)-g(b-1)]+\left[g(t)_t\Delta_{\rho(b)}^{-\mu}f(t)\right]_{t=a}^{t=b-1}-\sum_{t=a}^{b-2}g^\sigma(t)\Delta({}_t\Delta_{\rho(b)}^{-\mu}f(t))\\
&\quad=f(b-1)g(b)-f(a)g(a)-\frac{\mu}{\Gamma(\mu+1)}g(a)\sum_{s=\sigma(a)}^{b-1}(s+\mu-\sigma(a))^{(\mu-1)}f(s)\\
&\qquad+\sum_{t=a}^{b-2}\left({}_t\Delta_{\rho(b)}^{\alpha}f(t)\right)g^\sigma(t)\,,
\end{aligned}$$

where the first equality follows from the usual summation by parts formula. Putting this into (9), we get:

$$\begin{aligned}
&\sum_{t=a}^{b-1}f(t)_a\Delta_t^\alpha g(t)=f(b-1)g(b)-f(a)g(a)+\sum_{t=a}^{b-2}\left({}_t\Delta_{\rho(b)}^{\alpha}f(t)\right)g^\sigma(t)\\
&+\frac{g(a)\mu}{\Gamma(\mu+1)}\sum_{t=a}^{b-1}\frac{(t+\mu-a)^{(\mu-1)}}{\Gamma(\mu)}f(t)-\frac{g(a)\mu}{\Gamma(\mu+1)}\sum_{s=\sigma(a)}^{b-1}(s+\mu-\sigma(a))^{(\mu-1)}f(s).
\end{aligned}$$

The theorem is proved. □

3.2. **Necessary Optimality Conditions.** We begin to fix two arbitrary real numbers α and β such that $\alpha,\beta\in(0,1]$. Further, we put $\mu=1-\alpha$ and $\nu=1-\beta$.

Let a function $L(t,u,v,w):\mathbb{T}^\kappa\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be given. We assume that the second-order partial derivatives L_{uu}, L_{uv}, L_{uw}, L_{vw}, L_{vv}, and L_{ww} exist and are continuous.

Consider the functional $\mathcal{L}:\mathcal{F}\to\mathbb{R}$ defined by

$$\mathcal{L}(y(\cdot))=\sum_{t=a}^{b-1}L(t,y^\sigma(t),{}_a\Delta_t^\alpha y(t),{}_t\Delta_b^\beta y(t)) \tag{10}$$

and the problem, that we denote by (P), of minimizing (10) subject to the boundary conditions $y(a)=A$ and $y(b)=B$ ($A,B\in\mathbb{R}$). Our aim is to derive necessary conditions of first and second order for problem (P).

Definition 3.3. For $f\in\mathcal{F}$ we define the norm

$$\|f\|=\max_{t\in\mathbb{T}^\kappa}|f^\sigma(t)|+\max_{t\in\mathbb{T}^\kappa}|{}_a\Delta_t^\alpha f(t)|+\max_{t\in\mathbb{T}^\kappa}|{}_t\Delta_b^\beta f(t)|.$$

A function $\tilde{y}\in\mathcal{F}$ with $\tilde{y}(a)=A$ and $\tilde{y}(b)=B$ is called a local minimizer for problem (P) provided there exists $\delta>0$ such that $\mathcal{L}(\tilde{y})\le\mathcal{L}(y)$ for all $y\in\mathcal{F}$ with $y(a)=A$ and $y(b)=B$ and $\|y-\tilde{y}\|<\delta$.

Remark 4. It is easy to see that Definition 3.3 gives a norm in $\mathcal{F}$. Indeed, it is clear that $||f||$ is nonnegative, and for an arbitrary $f\in\mathcal{F}$ and $k\in\mathbb{R}$ we have

$\|kf\| = |k|\|f\|$. The triangle inequality is also easy to prove:

$$\begin{aligned}
\|f+g\| &= \max_{t\in\mathbb{T}^\kappa}|f(t)+g(t)| + \max_{t\in\mathbb{T}^\kappa}|{}_a\Delta_t^\alpha(f+g)(t)| + \max_{t\in\mathbb{T}^\kappa}|{}_t\Delta_b^\alpha(f+g)(t)|\\
&\le \max_{t\in\mathbb{T}^\kappa}[|f(t)|+|g(t)|] + \max_{t\in\mathbb{T}^\kappa}[|{}_a\Delta_t^\alpha f(t)| + |{}_a\Delta_t^\alpha g(t)|]\\
&\quad + \max_{t\in\mathbb{T}^\kappa}[|{}_t\Delta_b^\alpha f(t)| + |{}_t\Delta_b^\alpha g(t)|]\\
&\le \|f\|+\|g\|.
\end{aligned}$$

The only possible doubt is to prove that $||f|| = 0$ implies that $f(t) = 0$ for any $t \in \mathbb{T} = \{a, a+1, \ldots, b\}$. Suppose $||f|| = 0$. It follows that

$$\max_{t\in\mathbb{T}^\kappa}|f^\sigma(t)| = 0\,, \tag{11}$$

$$\max_{t\in\mathbb{T}^\kappa}|{}_a\Delta_t^\alpha f(t)| = 0\,, \tag{12}$$

$$\max_{t\in\mathbb{T}^\kappa}|{}_t\Delta_b^\beta f(t)| = 0\,. \tag{13}$$

From (11) we conclude that $f(t) = 0$ for all $t \in \{a+1, \ldots, b\}$. It remains to prove that $f(a) = 0$. To prove this we use (12) (or (13)). Indeed, from (11) we can write

$$\begin{aligned}
{}_a\Delta_t^\alpha f(t) &= \Delta\left(\frac{1}{\Gamma(1-\alpha)}\sum_{s=a}^{t}(t+1-\alpha-\sigma(s))^{(-\alpha)}f(s)\right)\\
&= \frac{1}{\Gamma(1-\alpha)}\left(\sum_{s=a}^{t+1}(t+2-\alpha-\sigma(s))^{(-\alpha)}f(s) - \sum_{s=a}^{t}(t+1-\alpha-\sigma(s))^{(-\alpha)}f(s)\right)\\
&= \frac{1}{\Gamma(1-\alpha)}\left((t+2-\alpha-\sigma(a))^{(-\alpha)}f(a) - (t+1-\alpha-\sigma(a))^{(-\alpha)}f(a)\right)\\
&= \frac{f(a)}{\Gamma(1-\alpha)}\Delta(t+1-\alpha-\sigma(a))^{(-\alpha)}
\end{aligned}$$

and since by (12) ${}_a\Delta_t^\alpha f(t) = 0$, one concludes that $f(a) = 0$ (because $(t+1-\alpha-\sigma(a))^{(-\alpha)}$ is not a constant).

Definition 3.4. A function $\eta \in \mathcal{F}$ is called an admissible variation for problem (P) provided $\eta \neq 0$ and $\eta(a) = \eta(b) = 0$.

The next theorem presents a first order necessary condition for problem (P).

Theorem 3.5 (The fractional discrete-time Euler–Lagrange equation)**.** *If $\tilde{y} \in \mathcal{F}$ is a local minimizer for problem (P), then*

$$L_u[\tilde{y}](t) + {}_t\Delta_{\rho(b)}^\alpha L_v[\tilde{y}](t) + {}_a\Delta_t^\beta L_w[\tilde{y}](t) = 0 \tag{14}$$

holds for all $t \in \mathbb{T}^{\kappa^2}$, where the operator $[\cdot]$ is defined by

$$[y](s) = (s, y^\sigma(s), {}_a\Delta_s^\alpha y(s), {}_s\Delta_b^\beta y(s)).$$

Proof. Suppose that $\tilde{y}(\cdot)$ is a local minimizer of $\mathcal{L}(\cdot)$. Let $\eta(\cdot)$ be an arbitrary fixed admissible variation and define the function $\Phi : \left(-\frac{\delta}{\|\eta(\cdot)\|}, \frac{\delta}{\|\eta(\cdot)\|}\right) \to \mathbb{R}$ by

$$\Phi(\varepsilon) = \mathcal{L}(\tilde{y}(\cdot) + \varepsilon\eta(\cdot)). \tag{15}$$

This function has a minimum at $\varepsilon = 0$, so we must have $\Phi'(0) = 0$, i.e.,

$$\sum_{t=a}^{b-1}\left[L_u[\tilde{y}](t)\eta^\sigma(t) + L_v[\tilde{y}](t){}_a\Delta_t^\alpha\eta(t) + L_w[\tilde{y}](t){}_t\Delta_b^\beta\eta(t)\right] = 0,$$

which we may write, equivalently, as

$$\begin{aligned}L_u[\tilde{y}](t)\eta^\sigma(t)|_{t=\rho(b)} &+ \sum_{t=a}^{b-2} L_u[\tilde{y}](t)\eta^\sigma(t)\\ &+ \sum_{t=a}^{b-1} L_v[\tilde{y}](t)_a\Delta_t^\alpha \eta(t) + \sum_{t=a}^{b-1} L_w[\tilde{y}](t)_t\Delta_b^\beta \eta(t) = 0. \quad (16)\end{aligned}$$

Using Theorem 3.2, and the fact that $\eta(a) = \eta(b) = 0$, we get for the third term in (16) that

$$\sum_{t=a}^{b-1} L_v[\tilde{y}](t)_a\Delta_t^\alpha \eta(t) = \sum_{t=a}^{b-2} \left({}_t\Delta_{\rho(b)}^\alpha L_v[\tilde{y}](t)\right) \eta^\sigma(t). \quad (17)$$

Using (8) it follows that

$$\begin{aligned}&\sum_{t=a}^{b-1} L_w[\tilde{y}](t)_t\Delta_b^\beta \eta(t)\\ &\quad = -\sum_{t=a}^{b-1} L_w[\tilde{y}](t)\Delta({}_t\Delta_b^{-\nu}\eta(t))\\ &\quad = -\sum_{t=a}^{b-1} L_w[\tilde{y}](t)\left[{}_t\Delta_{\rho(b)}^{-\nu}\Delta\eta(t) - \frac{\nu}{\Gamma(\nu+1)}(b+\nu-\sigma(t))^{(\nu-1)}\eta(b)\right]\\ &\quad = -\left(\sum_{t=a}^{b-1} L_w[\tilde{y}](t)_t\Delta_{\rho(b)}^{-\nu}\Delta\eta(t) - \frac{\nu\eta(b)}{\Gamma(\nu+1)}\sum_{t=a}^{b-1}(b+\nu-\sigma(t))^{(\nu-1)}L_w[\tilde{y}](t)\right). \quad (18)\end{aligned}$$

We now use Lemma 3.1 to get

$$\begin{aligned}&\sum_{t=a}^{b-1} L_w[\tilde{y}](t)_t\Delta_{\rho(b)}^{-\nu}\Delta\eta(t)\\ &\quad = \sum_{t=a}^{b-1} L_w[\tilde{y}](t)\Delta\eta(t) + \frac{\nu}{\Gamma(\nu+1)}\sum_{t=a}^{b-2} L_w[\tilde{y}](t)\sum_{s=\sigma(t)}^{b-1}(s+\nu-\sigma(t))^{(\nu-1)}\Delta\eta(s)\\ &\quad = \sum_{t=a}^{b-1} L_w[\tilde{y}](t)\Delta\eta(t) + \frac{\nu}{\Gamma(\nu+1)}\sum_{t=a}^{b-1}\Delta\eta(t)\sum_{s=a}^{t-1}(t+\nu-\sigma(s))^{(\nu-1)}L_w[\tilde{y}](s)\\ &\quad = \sum_{t=a}^{b-1}\Delta\eta(t)_a\Delta_t^{-\nu}L_w[\tilde{y}](t). \quad (19)\end{aligned}$$

We apply again the usual summation by parts formula, this time to (19), to obtain:

$$\begin{aligned}
&\sum_{t=a}^{b-1}\Delta\eta(t)_a\Delta_t^{-\nu}L_w[\tilde{y}](t)\\
&\qquad=\sum_{t=a}^{b-2}\Delta\eta(t)_a\Delta_t^{-\nu}L_w[\tilde{y}](t)+(\eta(b)-\eta(\rho(b)))_a\Delta_t^{-\nu}L_w[\tilde{y}](t)|_{t=\rho(b)}\\
&\qquad=\left[\eta(t)_a\Delta_t^{-\nu}L_w[\tilde{y}](t)\right]_{t=a}^{t=b-1}-\sum_{t=a}^{b-2}\eta^\sigma(t)\Delta({}_a\Delta_t^{-\nu}L_w[\tilde{y}](t))\\
&\qquad\quad+\eta(b)_a\Delta_t^{-\nu}L_w[\tilde{y}](t)|_{t=\rho(b)}-\eta(b-1)_a\Delta_t^{-\nu}L_w[\tilde{y}](t)|_{t=\rho(b)}\\
&\qquad=\eta(b)_a\Delta_t^{-\nu}L_w[\tilde{y}](t)|_{t=\rho(b)}-\eta(a)_a\Delta_t^{-\nu}L_w[\tilde{y}](t)|_{t=a}-\sum_{t=a}^{b-2}\eta^\sigma(t)_a\Delta_t^{\beta}L_w[\tilde{y}](t).
\end{aligned}\tag{20}$$

Since $\eta(a)=\eta(b)=0$ it follows, from (19) and (20), that

$$\sum_{t=a}^{b-1}L_w[\tilde{y}](t)_t\Delta_{\rho(b)}^{-\nu}\Delta\eta(t)=-\sum_{t=a}^{b-2}\eta^\sigma(t)_a\Delta_t^{\beta}L_w[\tilde{y}](t)$$

and, after inserting in (18), that

$$\sum_{t=a}^{b-1}L_w[\tilde{y}](t)_t\Delta_b^{\beta}\eta(t)=\sum_{t=a}^{b-2}\eta^\sigma(t)_a\Delta_t^{\beta}L_w[\tilde{y}](t).\tag{21}$$

By (17) and (21) we may write (16) as

$$\sum_{t=a}^{b-2}\left[L_u[\tilde{y}](t)+{}_t\Delta_{\rho(b)}^{\alpha}L_v[\tilde{y}](t)+{}_a\Delta_t^{\beta}L_w[\tilde{y}](t)\right]\eta^\sigma(t)=0.$$

Since the values of $\eta^\sigma(t)$ are arbitrary for $t\in\mathbb{T}^{\kappa^2}$, the Euler–Lagrange equation (14) holds along $\tilde{y}$. □

Remark 5. If the initial condition $y(a)=A$ is not present (i.e., $y(a)$ is free), we can use standard techniques to show that the following supplementary condition must be fulfilled:

$$\begin{aligned}
-L_v(a)+\frac{\mu}{\Gamma(\mu+1)}\Bigg(&\sum_{t=a}^{b-1}(t+\mu-a)^{(\mu-1)}L_v[\tilde{y}](t)\\
&-\sum_{t=\sigma(a)}^{b-1}(t+\mu-\sigma(a))^{(\mu-1)}L_v[\tilde{y}](t)\Bigg)+L_w(a)=0.
\end{aligned}\tag{22}$$

Similarly, if $y(b)=B$ is not present (i.e., $y(b)$ is free), the equality

$$\begin{aligned}
&L_u(\rho(b))+L_v(\rho(b))-L_w(\rho(b))\\
&\qquad+\frac{\nu}{\Gamma(\nu+1)}\Bigg(\sum_{t=a}^{b-1}(b+\nu-\sigma(t))^{(\nu-1)}L_w[\tilde{y}](t)\\
&\qquad\qquad-\sum_{t=a}^{b-2}(\rho(b)+\nu-\sigma(t))^{(\nu-1)}L_w[\tilde{y}](t)\Bigg)=0
\end{aligned}\tag{23}$$

holds. We just note that the first term in (23) arises from the first term on the left hand side of (16). Equalities (22) and (23) are the fractional discrete-time *natural boundary conditions.*

The next result is a particular case of our Theorem 3.5.

Corollary 1 (The discrete-time Euler–Lagrange equation – cf., e.g., [12, 14])**.** *If $\tilde{y}$ is a solution to the problem*

$$\begin{gathered}\mathcal{L}(y(\cdot)) = \sum_{t=a}^{b-1} L(t, y(t+1), \Delta y(t)) \longrightarrow \min \\ y(a) = A\,, \quad y(b) = B\,,\end{gathered} \tag{24}$$

then $L_u(t, \tilde{y}(t+1), \Delta\tilde{y}(t)) - \Delta L_v(t, \tilde{y}(t+1), \Delta\tilde{y}(t)) = 0$ for all $t \in \{a, \ldots, b-2\}$.

Proof. Follows from Theorem 3.5 with $\alpha = 1$ and a L not depending on w. □

We derive now the second order necessary condition for problem (P), i.e., we obtain Legendre's necessary condition for the fractional difference setting.

Theorem 3.6 (The fractional discrete-time Legendre condition)**.** *If $\tilde{y} \in \mathcal{F}$ is a local minimizer for problem (P), then the inequality*

$$\begin{aligned}
&L_{uu}[\tilde{y}](t) + 2L_{uv}[\tilde{y}](t) + L_{vv}[\tilde{y}](t) + L_{vv}[\tilde{y}](\sigma(t))(\mu-1)^2 \\
&\quad + \sum_{s=\sigma(\sigma(t))}^{b-1} L_{vv}[\tilde{y}](s)\left(\frac{\mu(\mu-1)\prod_{i=0}^{s-t-3}(\mu+i+1)}{(s-t)\Gamma(s-t)}\right)^2 + 2L_{uw}[\tilde{y}](t)(\nu-1) \\
&\qquad + 2(\nu-1)L_{vw}[\tilde{y}](t) + 2(\mu-1)L_{vw}[\tilde{y}](\sigma(t)) + L_{ww}[\tilde{y}](t)(1-\nu)^2 \\
&\qquad\quad + L_{ww}[\tilde{y}](\sigma(t)) + \sum_{s=a}^{t-1} L_{ww}[\tilde{y}](s)\left(\frac{\nu(1-\nu)\prod_{i=0}^{t-s-2}(\nu+i)}{(\sigma(t)-s)\Gamma(\sigma(t)-s)}\right)^2 \geq 0
\end{aligned}$$

holds for all $t \in \mathbb{T}^{\kappa^2}$, where $[\tilde{y}](t) = (t, \tilde{y}^\sigma(t), {}_a\Delta_t^\alpha \tilde{y}(t), {}_t\Delta_b^\beta \tilde{y}(t))$.

Proof. By the hypothesis of the theorem, and letting Φ be as in (15), we get

$$\Phi''(0) \geq 0 \tag{25}$$

for an arbitrary admissible variation $\eta(\cdot)$. Inequality (25) is equivalent to

$$\begin{aligned}
&\sum_{t=a}^{b-1} \Big[L_{uu}[\tilde{y}](t)(\eta^\sigma(t))^2 + 2L_{uv}[\tilde{y}](t)\eta^\sigma(t)_a\Delta_t^\alpha\eta(t) + L_{vv}[\tilde{y}](t)({}_a\Delta_t^\alpha\eta(t))^2 \\
&+2L_{uw}[\tilde{y}](t)\eta^\sigma(t)_t\Delta_b^\beta\eta(t) + 2L_{vw}[\tilde{y}](t)_a\Delta_t^\alpha\eta(t)_t\Delta_b^\beta\eta(t) + L_{ww}[\tilde{y}](t)({}_t\Delta_b^\beta\eta(t))^2\Big] \geq 0.
\end{aligned}$$

Let $\tau \in \mathbb{T}^{\kappa^2}$ be arbitrary and define $\eta : \mathbb{T} \to \mathbb{R}$ by

$$\eta(t) = \begin{cases} 1 & \text{if } t = \sigma(\tau); \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\eta(a) = \eta(b) = 0$, i.e., η is an admissible variation. Using (7) (note that $\eta(a) = 0$), we get

$$\begin{aligned}
&\sum_{t=a}^{b-1}\left[L_{uu}[\tilde{y}](t)(\eta^\sigma(t))^2 + 2L_{uv}[\tilde{y}](t)\eta^\sigma(t)_a\Delta_t^\alpha\eta(t) + L_{vv}[\tilde{y}](t)({}_a\Delta_t^\alpha\eta(t))^2\right]\\
&\quad= \sum_{t=a}^{b-1}\Big\{L_{uu}[\tilde{y}](t)(\eta^\sigma(t))^2\\
&\qquad +2L_{uv}[\tilde{y}](t)\eta^\sigma(t)\left[\Delta\eta(t) + \frac{\mu}{\Gamma(\mu+1)}\sum_{s=a}^{t-1}(t+\mu-\sigma(s))^{(\mu-1)}\Delta\eta(s)\right]\\
&\qquad +L_{vv}[\tilde{y}](t)\left(\Delta\eta(t) + \frac{\mu}{\Gamma(\mu+1)}\sum_{s=a}^{t-1}(t+\mu-\sigma(s))^{(\mu-1)}\Delta\eta(s)\right)^2\Big\}\\
&\quad= L_{uu}[\tilde{y}](\tau) + 2L_{uv}[\tilde{y}](\tau) + L_{vv}[\tilde{y}](\tau)\\
&\qquad + \sum_{t=\sigma(\tau)}^{b-1} L_{vv}[\tilde{y}](t)\left(\Delta\eta(t) + \frac{\mu}{\Gamma(\mu+1)}\sum_{s=a}^{t-1}(t+\mu-\sigma(s))^{(\mu-1)}\Delta\eta(s)\right)^2.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\sum_{t=\sigma(\sigma(\tau))}^{b-1} L_{vv}[\tilde{y}](t)\left(\frac{\mu}{\Gamma(\mu+1)}\sum_{s=a}^{t-1}(t+\mu-\sigma(s))^{(\mu-1)}\Delta\eta(s)\right)^2 + L_{vv}(\sigma(\tau))(-1+\mu)^2\\
&\quad= \sum_{t=\sigma(\tau)}^{b-1} L_{vv}[\tilde{y}](t)\left(\Delta\eta(t) + \frac{\mu}{\Gamma(\mu+1)}\sum_{s=a}^{t-1}(t+\mu-\sigma(s))^{(\mu-1)}\Delta\eta(s)\right)^2.
\end{aligned}$$

We show next that

$$\begin{aligned}
&\sum_{t=\sigma(\sigma(\tau))}^{b-1} L_{vv}[\tilde{y}](t)\left(\frac{\mu}{\Gamma(\mu+1)}\sum_{s=a}^{t-1}(t+\mu-\sigma(s))^{(\mu-1)}\Delta\eta(s)\right)^2\\
&\qquad\qquad= \sum_{t=\sigma(\sigma(\tau))}^{b-1} L_{vv}[\tilde{y}](t)\left(\frac{\mu(\mu-1)\prod_{i=0}^{t-\tau-3}(\mu+i+1)}{(t-\tau)\Gamma(t-\tau)}\right)^2.
\end{aligned}$$

Let $t \in [\sigma(\sigma(\tau)), b-1] \cap \mathbb{Z}$. Then,

$$\begin{aligned}
&\frac{\mu}{\Gamma(\mu+1)}\sum_{s=a}^{t-1}(t+\mu-\sigma(s))^{(\mu-1)}\Delta\eta(s)\\
&\quad= \frac{\mu}{\Gamma(\mu+1)}\left[\sum_{s=a}^{\tau}(t+\mu-\sigma(s))^{(\mu-1)}\Delta\eta(s) + \sum_{s=\sigma(\tau)}^{t-1}(t+\mu-\sigma(s))^{(\mu-1)}\Delta\eta(s)\right]\\
&\quad= \frac{\mu}{\Gamma(\mu+1)}\left[(t+\mu-\sigma(\tau))^{(\mu-1)} - (t+\mu-\sigma(\sigma(\tau)))^{(\mu-1)}\right]\\
&\quad= \frac{\mu}{\Gamma(\mu+1)}\left[\frac{\Gamma(t+\mu-\sigma(\tau)+1)}{\Gamma(t+\mu-\sigma(\tau)+1-(\mu-1))} - \frac{\Gamma(t+\mu-\sigma(\sigma(\tau))+1)}{\Gamma(t+\mu-\sigma(\sigma(\tau))+1-(\mu-1))}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu}{\Gamma(\mu+1)}\left[\frac{\Gamma(t+\mu-\tau)}{\Gamma(t-\tau+1)} - \frac{\Gamma(t-\tau+\mu-1)}{\Gamma(t-\tau)}\right] \\
&= \frac{\mu}{\Gamma(\mu+1)}\left[\frac{(t+\mu-\tau-1)\Gamma(t+\mu-\tau-1)}{(t-\tau)\Gamma(t-\tau)} - \frac{(t-\tau)\Gamma(t-\tau+\mu-1)}{(t-\tau)\Gamma(t-\tau)}\right] \\
&= \frac{\mu}{\Gamma(\mu+1)}\frac{(\mu-1)\Gamma(t-\tau+\mu-1)}{(t-\tau)\Gamma(t-\tau)} \\
&= \frac{\mu(\mu-1)\prod_{i=0}^{t-\tau-3}(\mu+i+1)}{(t-\tau)\Gamma(t-\tau)},
\end{aligned} \tag{26}
$$

which proves our claim. Observe that we can write ${}_t\Delta_b^{\beta}\eta(t) = -{}_t\Delta_{\rho(b)}^{-\nu}\Delta\eta(t)$ since $\eta(b)=0$. It is not difficult to see that the following equality holds:

$$
\sum_{t=a}^{b-1} 2L_{uw}[\tilde{y}](t)\eta^{\sigma}(t){}_t\Delta_b^{\beta}\eta(t) = -\sum_{t=a}^{b-1} 2L_{uw}[\tilde{y}](t)\eta^{\sigma}(t){}_t\Delta_{\rho(b)}^{-\nu}\Delta\eta(t) = 2L_{uw}[\tilde{y}](\tau)(\nu-1).
$$

Moreover,

$$
\begin{aligned}
&\sum_{t=a}^{b-1} 2L_{vw}[\tilde{y}](t){}_a\Delta_t^{\alpha}\eta(t){}_t\Delta_b^{\beta}\eta(t) \\
&\quad = -2\sum_{t=a}^{b-1} L_{vw}[\tilde{y}](t)\left\{\left(\Delta\eta(t) + \frac{\mu}{\Gamma(\mu+1)}\cdot\sum_{s=a}^{t-1}(t+\mu-\sigma(s))^{(\mu-1)}\Delta\eta(s)\right)\right. \\
&\qquad \left.\cdot\left[\Delta\eta(t) + \frac{\nu}{\Gamma(\nu+1)}\sum_{s=\sigma(t)}^{b-1}(s+\nu-\sigma(t))^{(\nu-1)}\Delta\eta(s)\right]\right\} \\
&\quad = 2(\nu-1)L_{vw}[\tilde{y}](\tau) + 2(\mu-1)L_{vw}[\tilde{y}](\sigma(\tau)).
\end{aligned}
$$

Finally, we have that

$$
\begin{aligned}
&\sum_{t=a}^{b-1} L_{ww}[\tilde{y}](t)({}_t\Delta_b^{\beta}\eta(t))^2 \\
&\quad = \sum_{t=a}^{\sigma(\tau)} L_{ww}[\tilde{y}](t)\left[\Delta\eta(t) + \frac{\nu}{\Gamma(\nu+1)}\sum_{s=\sigma(t)}^{b-1}(s+\nu-\sigma(t))^{(\nu-1)}\Delta\eta(s)\right]^2 \\
&\quad = \sum_{t=a}^{\tau-1} L_{ww}[\tilde{y}](t)\left[\frac{\nu}{\Gamma(\nu+1)}\sum_{s=\sigma(t)}^{b-1}(s+\nu-\sigma(t))^{(\nu-1)}\Delta\eta(s)\right]^2 \\
&\qquad + L_{ww}[\tilde{y}](\tau)(1-\nu)^2 + L_{ww}[\tilde{y}](\sigma(\tau)) \\
&\quad = \sum_{t=a}^{\tau-1} L_{ww}[\tilde{y}](t)\left[\frac{\nu}{\Gamma(\nu+1)}\left\{(\tau+\nu-\sigma(t))^{(\nu-1)} - (\sigma(\tau)+\nu-\sigma(t))^{(\nu-1)}\right\}\right]^2 \\
&\qquad + L_{ww}[\tilde{y}](\tau)(1-\nu)^2 + L_{ww}[\tilde{y}](\sigma(\tau)).
\end{aligned}
$$

Similarly as we have done in (26), we obtain that

$$
\frac{\nu}{\Gamma(\nu+1)}\left[(\tau+\nu-\sigma(t))^{(\nu-1)} - (\sigma(\tau)+\nu-\sigma(t))^{(\nu-1)}\right] = \frac{\nu(1-\nu)\prod_{i=0}^{\tau-t-2}(\nu+i)}{(\sigma(\tau)-t)\Gamma(\sigma(\tau)-t)}.
$$

We are done with the proof. □

A trivial corollary of our result gives the discrete-time version of Legendre's necessary condition.

Corollary 2 (The discrete-time Legendre condition – cf., e.g., [12, 17])**.** *If $\tilde{y}$ is a solution to the problem (24), then*

$$L_{uu}[\tilde{y}](t) + 2L_{uv}[\tilde{y}](t) + L_{vv}[\tilde{y}](t) + L_{vv}[\tilde{y}](\sigma(t)) \geq 0$$

holds for all $t \in \mathbb{T}^{\kappa^2}$, where $[\tilde{y}](t) = (t, \tilde{y}^\sigma(t), \Delta\tilde{y}(t))$.

Proof. We consider problem (P) with $\alpha = 1$ and L not depending on w. The choice $\alpha = 1$ implies $\mu = 0$, and the result follows immediately from Theorem 3.6. ☐

4. Examples.

In this section we present three illustrative examples. The results were obtained using the open source Computer Algebra System Maxima.[1] All computations were done running Maxima on an Intel® Core^{TM}2 Duo, CPU of 2.27GHz with 3Gb of RAM. Our Maxima definitions are given in Appendix.

Example 1. Let us consider the following problem:

$$J_\alpha(y) = \sum_{t=0}^{b-1} \left({}_0\Delta_t^\alpha y(t)\right)^2 \longrightarrow \min\,, \quad y(0) = A\,, \quad y(b) = B\,. \tag{27}$$

In this case Theorem 3.6 is trivially satisfied. We obtain the solution $\tilde{y}$ to our Euler-Lagrange equation (14) for the case $b = 2$ using the computer algebra system Maxima. Using our Maxima package (see the definition of the command `extremal` in Appendix) we do

```
L1:v^2$
extremal(L1,0,2,A,B,alpha,alpha);
```

to obtain (2 seconds)

$$\tilde{y}(1) = \frac{2\,\alpha\,B + \left(\alpha^3 - \alpha^2 + 2\,\alpha\right)\,A}{2\,\alpha^2 + 2}\,. \tag{28}$$

For the particular case $\alpha = 1$ the equality (28) gives $\tilde{y}(1) = \frac{A+B}{2}$, which coincides with the solution to the (non-fractional) discrete problem

$$\sum_{t=0}^{1} \left(\Delta y(t)\right)^2 = \sum_{t=0}^{1} \left(y(t+1) - y(t)\right)^2 \longrightarrow \min\,, \quad y(0) = A\,, \quad y(2) = B\,.$$

Similarly, we can obtain exact formulas of the extremal on bigger intervals (for bigger values of b). For example, the solution of problem (27) with $b = 3$ is (35 seconds)

$$\begin{aligned}
\tilde{y}(1) &= \frac{\left(6\,\alpha^2 + 6\,\alpha\right)\,B + \left(2\,\alpha^5 + 2\,\alpha^4 + 10\,\alpha^3 - 2\,\alpha^2 + 12\,\alpha\right)\,A}{3\,\alpha^4 + 6\,\alpha^3 + 15\,\alpha^2 + 12}\,,\\
\tilde{y}(2) &= \frac{\left(12\,\alpha^3 + 12\,\alpha^2 + 24\,\alpha\right)\,B + \left(\alpha^6 + \alpha^5 + 7\,\alpha^4 - \alpha^3 + 4\,\alpha^2 + 12\,\alpha\right)\,A}{6\,\alpha^4 + 12\,\alpha^3 + 30\,\alpha^2 + 24}\,;
\end{aligned}$$

[1] http://maxima.sourceforge.net

and the solution of problem (27) with $b = 4$ is (72 seconds)

$$\begin{aligned}
\tilde{y}(1) &= \frac{3\,\alpha^7 + 15\,\alpha^6 + 57\,\alpha^5 + 69\,\alpha^4 + 156\,\alpha^3 - 12\,\alpha^2 + 144\,\alpha}{\xi} A \\
&\quad + \frac{24\,\alpha^3 + 72\,\alpha^2 + 48\,\alpha}{\xi} B\,, \\
\tilde{y}(2) &= \frac{\alpha^8 + 5\,\alpha^7 + 22\,\alpha^6 + 32\,\alpha^5 + 67\,\alpha^4 + 35\,\alpha^3 + 54\,\alpha^2 + 72\,\alpha}{\xi} A \\
&\quad + \frac{24\,\alpha^4 + 72\,\alpha^3 + 120\,\alpha^2 + 72\,\alpha}{\xi} B\,, \\
\tilde{y}(3) &= \frac{\alpha^9 + 6\,\alpha^8 + 30\,\alpha^7 + 60\,\alpha^6 + 117\,\alpha^5 + 150\,\alpha^4 - 4\,\alpha^3 + 216\,\alpha^2 + 288\,\alpha}{\zeta} A \\
&\quad + \frac{72\,\alpha^5 + 288\,\alpha^4 + 792\,\alpha^3 + 576\,\alpha^2 + 864\,\alpha}{\zeta} B\,,
\end{aligned}$$

where

$$\begin{aligned}
\xi &= 4\,\alpha^6 + 24\,\alpha^5 + 88\,\alpha^4 + 120\,\alpha^3 + 196\,\alpha^2 + 144\,, \\
\zeta &= 24\,\alpha^6 + 144\,\alpha^5 + 528\,\alpha^4 + 720\,\alpha^3 + 1176\,\alpha^2 + 864\,.
\end{aligned}$$

Consider now problem (27) with $b = 4$, $A = 0$, and $B = 1$. In Table 1 we show the extremal values $\tilde{y}(1)$, $\tilde{y}(2)$, $\tilde{y}(3)$, and corresponding $\tilde{J}_\alpha$, for some values of α. Our numerical results show that the fractional extremal converges to the classical (integer order) extremal when α tends to one. This is illustrated in Figure 1. The numerical results from Table 1 and Figure 2 show that for this problem the smallest value of $\tilde{J}_\alpha$, $\alpha \in]0, 1]$, occur for $\alpha = 1$ (i.e., the smallest value of $\tilde{J}_\alpha$ occurs for the classical non-fractional case).

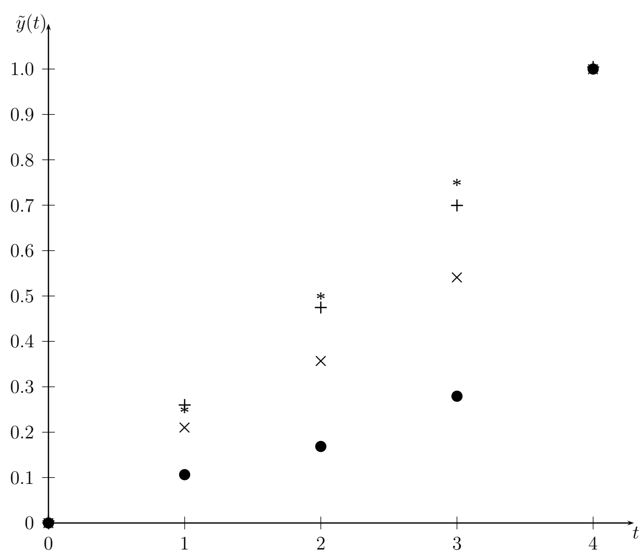

FIGURE 1. Extremal $\tilde{y}(t)$ of Example 1 with $b = 4$, $A = 0$, $B = 1$, and different α's ($\bullet$: $\alpha = 0.25$; $\times$: $\alpha = 0.5$; $+$: $\alpha = 0.75$; $*$: $\alpha = 1$).

α	$\tilde{y}(1)$	$\tilde{y}(2)$	$\tilde{y}(3)$	$\tilde{J}_\alpha$
0.25	0.10647146897355	0.16857982587479	0.2792657904952	0.90855653524095
0.50	0.20997375328084	0.35695538057743	0.54068241469816	0.67191601049869
0.75	0.25543605027861	0.4702345471038	0.69508876506414	0.4246209666969
1	0.25	0.5	0.75	0.25

TABLE 1. The extremal values $\tilde{y}(1)$, $\tilde{y}(2)$ and $\tilde{y}(3)$ of problem (27) with $b = 4$, $A = 0$, and $B = 1$ for different α's.

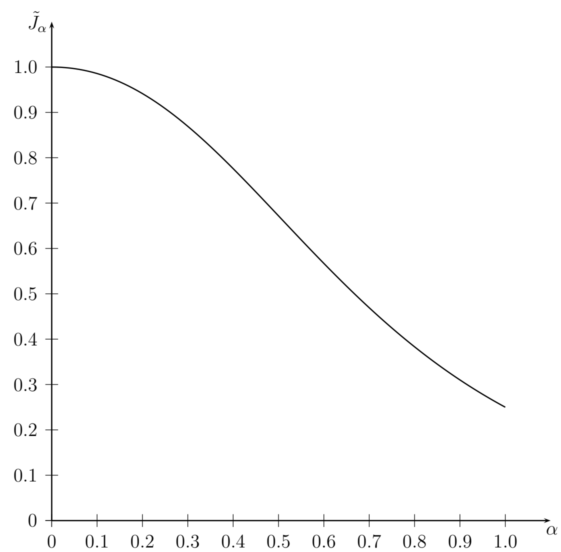

FIGURE 2. Function $\tilde{J}_\alpha$ of Example 1 with $b = 4$, $A = 0$, and $B = 1$.

Example 2. In this example we generalize problem (27) to

$$J_{\alpha,\beta} = \sum_{t=0}^{b-1} \gamma_1 \left({}_0\Delta_t^\alpha y(t)\right)^2 + \gamma_2 \left({}_t\Delta_b^\beta y(t)\right)^2 \longrightarrow \min\,, \quad y(0) = A\,, \quad y(b) = B\,. \tag{29}$$

As before, we solve the associated Euler-Lagrange equation (14) for the case $b = 2$ with the help of our Maxima package (35 seconds):

```
L2:(gamma[1])*v^2+(gamma[2])*w^2$
extremal(L2,0,2,A,B,alpha,beta);
```

$$\tilde{y}(1) = \frac{\left(2\,\gamma_2\,\beta + \gamma_1\,\alpha^3 - \gamma_1\,\alpha^2 + 2\,\gamma_1\,\alpha\right)\,A + \left(\gamma_2\,\beta^3 - \gamma_2\,\beta^2 + 2\,\gamma_2\,\beta + 2\,\gamma_1\,\alpha\right)\,B}{2\,\gamma_2\,\beta^2 + 2\,\gamma_1\,\alpha^2 + 2\,\gamma_2 + 2\,\gamma_1}\,.$$

Consider now problem (29) with $\gamma_1 = \gamma_2 = 1$, $b = 2$, $A = 0$, $B = 1$, and $\beta = \alpha$. In Table 2 we show the values of $\tilde{y}(1)$ and $\tilde{J}_\alpha := J_{\alpha,\alpha}(\tilde{y}(1))$ for some values of α. We concluded, numerically, that the fractional extremal $\tilde{y}(1)$ tends to the classical (non-fractional) extremal when α tends to one. Differently from Example 1, the smallest value of $\tilde{J}_\alpha$, $\alpha \in]0, 1]$, does not occur here for $\alpha = 1$ (see Figure 3). The smallest value of $\tilde{J}_\alpha$, $\alpha \in]0, 1]$, occurs for $\alpha = 0.61747447161482$.

α	$\tilde{y}(1)$	$\tilde{J}_\alpha$
0.25	0.22426470588235	0.96441291360294
0.50	0.375	0.9140625
0.75	0.4575	0.91720703125
1	0.5	1

TABLE 2. The extremal $\tilde{y}(1)$ of problem (29) for different values of α ($\gamma_1 = \gamma_2 = 1$, $b = 2$, $A = 0$, $B = 1$, and $\beta = \alpha$).

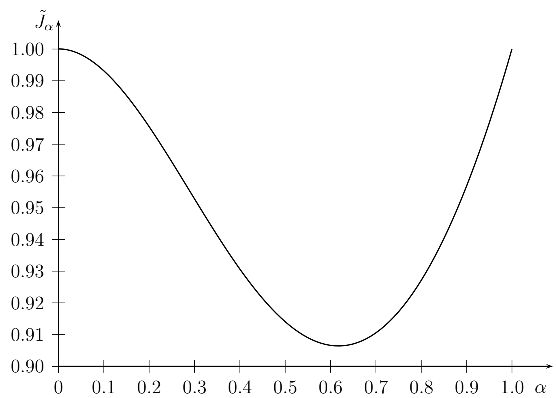

FIGURE 3. Function $\tilde{J}_\alpha$ of Example 2 with $\gamma_1 = \gamma_2 = 1$, $b = 2$, $A = 0$, $B = 1$, and $\beta = \alpha$.

Example 3. Our last example is a discrete version of the fractional continuous problem [4, Example 2]:

$$J_\alpha = \sum_{t=0}^{1} \frac{1}{2} \left({}_0\Delta_t^\alpha y(t)\right)^2 - y^\sigma(t) \longrightarrow \min\,, \quad y(0) = 0\,, \quad y(2) = 0\,. \tag{30}$$

The Euler-Lagrange extremal of (30) is easily obtained with our Maxima package (4 seconds):

```
L3:(1/2)*v^2-u;$
extremal(L3,0,2,0,0,alpha,beta);
```

$$\tilde{y}(1) = \frac{1}{\alpha^2 + 1}\,. \tag{31}$$

For the particular case $\alpha = 1$ the equality (31) gives $\tilde{y}(1) = \frac{1}{2}$, which coincides with the solution to the non-fractional discrete problem

$$\sum_{t=0}^{1} \frac{1}{2} \left(\Delta y(t)\right)^2 - y^\sigma(t) = \sum_{t=0}^{1} \frac{1}{2} \left(y(t+1) - y(t)\right)^2 - y(t+1) \longrightarrow \min\,,$$
$$y(0) = 0\,, \quad y(2) = 0\,.$$

In Table 3 we show the values of $\tilde{y}(1)$ and $\tilde{J}_\alpha$ for some α's. As seen in Figure 4, for $\alpha = 1$ one gets the maximum value of $\tilde{J}_\alpha$, $\alpha \in]0, 1]$.

5. **Conclusion.** The discrete-time calculus is a very important tool in practical applications and in the modeling of real phenomena. Therefore, it is not a surprise that fractional discrete calculus is recently under strong development. Possible

α	$\tilde{y}(1)$	$\tilde{J}_\alpha$
0.25	0.94117647058824	-0.47058823529412
0.50	0.8	-0.4
0.75	0.64	-0.32
1	0.5	-0.25

TABLE 3. Extremal values $\tilde{y}(1)$ of (30) for different α's

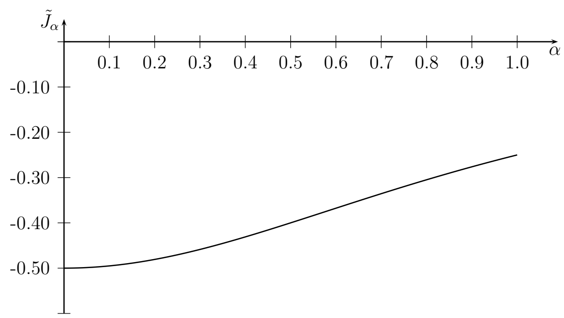

FIGURE 4. Function $\tilde{J}_\alpha$ of Example 3.

areas of application include the signal processing, where fractional derivatives of a discrete-time signal are particularly useful to describe noise processes [25].

In this paper we introduce the study of fractional discrete-time problems of the calculus of variations or order α, $0 < \alpha \leq 1$, with left and right discrete operators of Riemann–Liouville type. For $\alpha = 1$ we obtain the classical discrete-time results of the calculus of variations [19]. Main results of the paper include a fractional summation by parts formula (Theorem 3.2), a fractional discrete-time Euler–Lagrange equation (Theorem 3.5), transversality conditions (22) and (23), and a fractional discrete-time Legendre condition (Theorem 3.6). From the analysis of the results obtained from computer experiments, we conclude that when the value of α approaches one, the optimal value of the fractional discrete functional converges to the optimal value of the classical (non-fractional) discrete problem. On the other hand, the value of α for which the functional attains its minimum varies with the concrete problem under consideration.

This research is in its beginning phase, and it will be developed further in the future. Indeed, being the first work on fractional difference variational problems, much remains to be done. For example, one can extend the present results for higher-order problems of the calculus of variations with fractional discrete derivatives of any order. Moreover, our work also opens new possibilities of research for fractional continuous variational problems. In particular, to prove a fractional continuous Legendre necessary optimality condition, analogous to the fractional discrete result given by Theorem 3.6, is a stimulating open question. One of the referees called our attention to the fact that the initial conditions considered in this work have ARMA formats. The problem of finding a general formulation leading to the specification of the ARMA parameters seems to be also an interesting question.

Appendix. The following Maxima code implements Theorem 3.5. Examples illustrating the use of our procedure `extremal` are found in Section 4.

```
kill(all)$
ratprint:false$
simpsum:true$
tlimswitch:true$

sigma(t):=t+1$

rho(t):=t-1$

rho2(t):=rho(rho(t))$

Delta(exp,t):=block( define(f12(t),exp),
return((f12(sigma(t))-f12(t))) )$

p(x,y):=(gamma(x+1))/(gamma(x+1-y))$

SumL(a,t,nu,exp):=block(
define(f1(x),exp),
f1(t)+nu/gamma(nu+1)*sum((p(t+nu-sigma(r),nu-1))*f1(r),r,(a),((t-1)))
)$

SumR(t1,b,nu1,exp1):=block(
define(f2(x),exp1),
f2(t1)+nu1/gamma(nu1+1)*sum((p(s+nu1-sigma(t1),nu1-1))*f2(s),s,(t1+1),b)
)$

DeltaL(a2,t2,alpha2,exp2):=block(
[alpha1:ratsimp(alpha2),a:ratsimp(a2),t:ratsimp(t2)],
define(f3(x),exp2), define(q(x),SumL(a,x,1-alpha1,f3(x))),
q0:q(o),
o4:float(ev(ratsimp(Delta(q0,o)),nouns)), o41:subst(o=t,o4),
remfunction(f), remfunction(q), return(o41)
)$

DeltaR(t3,b,alpha3,exp3):=block(
[alpha1:ratsimp(alpha3),b:ratsimp(b),t:ratsimp(t3)],
define(f4(x),exp3), define(q1(o),(SumR(x,b,1-alpha1,f4(x)))),
q10:q1(z),
o5:float(ev(ratsimp(-Delta((SumR(x,b,1-alpha1,f4(x))),x)),nouns)),
o51:subst(x=t,o5), remfunction(f), remfunction(q1), return(o51)
)$

EL(exp7,a,b,alpha7,beta7):=block(
[a:ratsimp(a),b:ratsimp(b),alpha:ratsimp(alpha7),beta:ratsimp(beta7)],
define(LL(t,u,v,w),exp7), b1:diff(LL(t,u,v,w),u),
sa:subst([u=y(sigma(t)),v=DeltaL(a,t,alpha,y(o)),
                        w=DeltaR(t,b,beta,y(o))],b1),
b2:diff(LL(t,u,v,w),v),
sb:subst([t=x,u=y(sigma(x)),v=DeltaL(a,x,alpha,y(x)),
                             w=DeltaR(x,b,beta,y(x))],b2),
sb1:DeltaR(o,rho(b),alpha,sb), sb11:subst(o=t,sb1),
b3:diff(LL(t,u,v,w),w),
```

```
sc:subst([t=x,u=y(sigma(x)),v=DeltaL(a,x,alpha,y(x)),
                            w=DeltaR(x,b,beta,y(x))],b3),
sc2:DeltaL(a,p2,beta,sc), sc22:subst(p2=t,sc2),
return(sa+sb11+sc22)
)$

ELt(exp8,a,b,alpha8,beta8,t8):=
ratsimp(subst(t=t8,EL(exp8,a,b,alpha8,beta8)))$

extremal(L,a,b,A9,B9,alpha9,beta9):=block(
[a:ratsimp(a),b:ratsimp(b),alpha:ratsimp(alpha9),
beta:ratsimp(beta9),A1:ratsimp(A9), B1:ratsimp(B9)],
eqs:makelist(ratsimp(ELt(L,a,b,alpha,beta,a+i)),i,0,ratsimp((rho2(b)-a))),
vars:makelist(y(ratsimp(a+i)),i,1,ratsimp((rho(b)-a))),
Xi:[a],Xf:[b],Yi:[A1],Yf:[B1],
X:makelist(ratsimp(i),i,1,ratsimp((rho(b)-a))), X:append(Xi,X,Xf),
sols:algsys(subst([y(a)=A1,y(b)=B1],eqs),vars),
Y:makelist(rhs(sols[1][i]),i,1,ratsimp((rho(b)-a))),
Y:append(Yi,Y,Yf),
return(makegamma(ratsimp(minfactorial(makefact(sols[1])))))
)$
```

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